Algebraic topology

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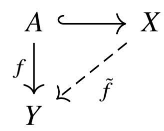
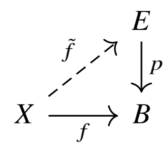
1.1 Homotopy

Two of the most basic questions homotopy theory attempts to answer are *extension problems* and *lifting problems*:

1. Extension problem: If A embeds in a topological space X, when can we extend a continuous function $f : A \to Y$ to a continuous function $\tilde{f} : X \to Y$?



2. Lifting problem: If $p : E \to B$ is a surjective map, when can we lift a continuous function $f : X \to B$ to a continuous $\tilde{f} : X \to E$?



We will answer these questions (in some cases) by using group theory to study the structure of continuous functions from X to Y.

The basic idea is that two continuous functions are homotopic if they can be continuously deformed to each other on their shared domain. Consider the following three curves γ_1, γ_2, and γ_3 in the punctured plane $\mathbb{C} \setminus \{0\}$. The curves γ_2 and γ_1 can be continuously deformed onto each other, but deforming either of them onto γ_3 would require passing through the origin, which is not in our domain.

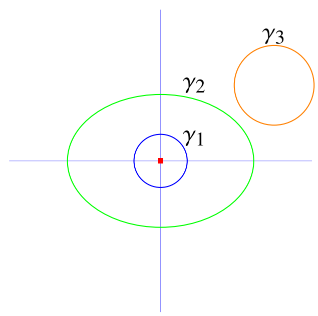


Figure 1: γ_1 and γ_2 are homotopy equivalent but γ_1 and γ_3 are not.

All functions we consider in this note will be continuous unless otherwise stated.

Definition 1.1. Two maps $f, g : X \to Y$ are homotopic relative to $A \subset X$ if f and g agree on A and there is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, $H(x, 1) = g(x)$, and $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$. We denote this as $f \simeq g \mathrm{rel} A$. H is called a *homotopy* from f to g, denoted by $H : f \to g$.

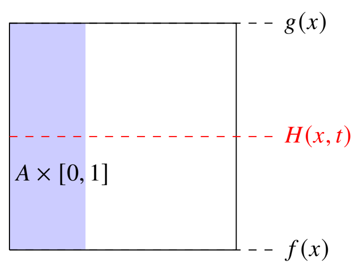


Figure 2: A visualization of the function H.

Example 1.2.

(i) In Figure 1, think of a closed curve γ as a map $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = \gamma(1)$. To apply Definition 1.1 to this example, let $X = [0, 1]$, $Y = \mathbb{C} \setminus \{0\}$ and $A = \{0, 1\}$.

(ii) Given a space X and any set $A \subset \mathbb{R}^n$, any two maps $f, g : \mathbb{R}^n \to \mathbb{R}^n$ that agree on A are homotopic. Intuitively, this is because we can get around the problem in Figure 1: there are no "holes" to prevent us from deforming closed curves onto each other. The easiest formal proof of this is to explicitly construct the homotopy by $H(x, t) = tg(x) + (1-t)f(x)$.

Next, we want to see that homotopy is an equivalence relation.

Proposition 1.3. *Fix $A \subset X$ and $j : A \to Y$. The relation $\simeq \mathrm{rel} A$ is an equivalence relation on the set of functions $\{f : X \to Y : f|_A = j\}$.*

Proof. Reflexivity is clear by letting $H(x, t) = f(x)$ for all $x \in X$ and $t \in [0, 1]$. For symmetry, given a homotopy $H_1 : f \to g$, we " reverse" it to get a homotopy $H_2 : g \to f$. Define $H_2(x, t) = H_2(x, 1 - t)$. Finally, let $H_1 : f \to g$ and $H_2 : g \to h$ be homotopies. We obtain a homotopy $H_3 : f \to h$ by "compressing" H_1 and H_2 together:

$$H_3(x,t) = \begin{cases} H_1(x, 2t), & 0 \le t \le 1/2, \\ H_2(x, 2t-1) & 1/2 \le t \le 1. \end{cases}$$

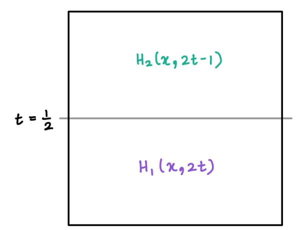


Figure 3: Compressing two homotopies to obtain a third.

□

These tricks of reversing and compressing homotopies to obtain new ones will show up several more times. Homotopy equivalence is also preserved by function composition.

Lemma 1.4. *Suppose we have maps between topological spaces*

$$(W, A) \underset{g'}{\overset{g}{\longrightarrow}} (X, B) \underset{h'}{\overset{h}{\longrightarrow}} (Y, C)$$

such that $g(A) = g'(A) = B$, $h(B) = h'(B) = C$, $g \simeq g' \mathrm{rel} A$ and $h \simeq g' \mathrm{rel} B$. Then, $h \circ g \simeq h' \circ g' \mathrm{rel} A$.

Proof. Let $G : g \to g'$ and $H : h \to h'$ be the homotopies. The homotopy $K : h \circ g \to h' \circ g'$ is defined naturally as a composition of H and G.

$$K(w, t) = H(G(w, t), t).$$

Check that (i) $K(w, 0) = h \circ g(w)$, (ii) $K(w, 1) = h' \circ g'(w)$, and (iii) for all $a \in A$, $K(a, t) = h \circ g(a) = h' \circ g'(a)$. □

Typically, we will only care about the case where A is a point.

Definition 1.5. A *pointed space* is a pair (X, x_0) where X is a topological space and x_0 a point in X. The point x_0 is referred to as the *basepoint*. If (X, x_0) and (Y, y_0) are pointed spaces, $f : X \to Y$ is a *pointed map* if $f(x_0) = y_0$.

Let $[X, Y]$ denote the homotopy equivalence classes of pointed maps from X to Y. It is convenient to omit the basepoint from our notation: we will see that the objects we study depend only on path-connected components, and not the basepoint.

Definition 1.6 (Homotopy groups)**.** The set $\pi_n(Y, y_0) = [S^n, Y]$, where the basepoint of S^n is $(1, 0, \cdots, 0)$ is the *nth homotopy group* of Y.

When $n = 1$, we call $\pi_1(Y, y_0)$, or simply $\pi_1(Y)$, the *fundamental group*.

1.2 The group structure of the fundamental group

It does not make sense to call an object a group unless it is one. The correct way to visualize $\pi_1(Y, y_0)$ is as homotopy equivalence classes of loops in Y that begin and end at y_0. In Figure 1, we said a closed curve was a map $\gamma : [0, 1] \to Y$ such that $\gamma(0) = \gamma(1)$. Instead of identifying the endpoints 0 and 1 in the *image* of γ, we can identify them in the *domain*. Identifying the endpoints of $[0, 1]$ gives us the circle S^1, so a closed curve, or a loop, is also a map $\gamma : S^1 \to Y$. This is a very useful correspondence and we will make use of both notions of the domain. For example, when we want to define a homotopy H between two loops, it will be convenient to think of the domain as $[0, 1]$. When we want to study properties of a loop f, we will think of the domain as S^1. (This is only a general heuristic: what domain we choose will depend heavily on what we are trying to achieve.)

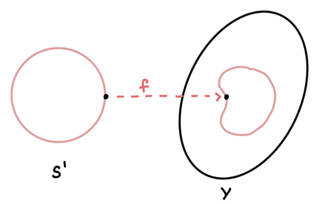


Figure 4: Each pointed map $f : S^1 \to Y$ is a loop in Y

Two loops are homotopy equivalent if one can be continuously deformed onto the other, like in Figure 1. We multiply elements of $\pi_1(Y, y_0)$ by concatenating the corresponding loops. The element $f \cdot g$ corresponds to first traversing the loop f, then g. Formally, thinking of f, g as maps $[0, 1] \to Y$

$$f \cdot g(s) = \begin{cases} f(2s), & 0 \leq s \leq 1/2 \\ g(2s-1), & 1/2 \leq s \leq 1. \end{cases}$$

As will often be the case with several of these operations,we have to check that this is well-defined: see Exercise 1.

Now we are ready to check that the fundamental group is, in fact, a group.

Theorem 1.7. *The fundamental group $\pi_1(Y, y_0)$ is a group under the operation $[f] \cdot [g] = [f \cdot g]$.*

Proof. *Step 1: the identity*

The identity element is the simplest map we can think of. Let $c : S^1 \to Y$ be the constant map $c(x) = y_0$ for all $x \in S^1$. Then, for any map $f : S^1 \to Y$, define a homotopy H from f to $c \cdot f$ by

$$H(s, t) = \begin{cases} y_0 & 2s \leq t, \\ f\left(\frac{2s-t}{2-t}\right) & 2s \geq t. \end{cases}$$

Check that $H(s, 0) = f(s)$ and $H(s, 1) = c \cdot f(s)$. How do you modify this construction to get a homotopoy H' from f to $f \cdot c$. This shows that c is a two-sided identity in $\pi_1(Y)$.

Step 2: associativity

I will present the homotopy showing that $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$, and leave the tedious task of verification to the reader.

$$H(s,t) = \begin{cases} f\left(\frac{4s}{2-t}\right) & 4s \le 2-t, \\ g(4s+t-2) & 2-t \le 4s \le 3-t, \\ h\left(\frac{4s+t-3}{1+t}\right) & 3-t \le 4s. \end{cases}$$

Step 3: inverses

Given $f : [0,1] \to Y$, define f^{-1} by $f^{-1}(s) = f(1-s)$. Intuitively, f^{-1} traverses the same loop as f, but in the opposite direction. Define a homotopy H from c to $f \cdot f^{-1}$,

$$H(s,t) = \begin{cases} f(2st) & 0 \le s \le 1/2 \\ f(2(1-s)t) & 1/s \le s \le 1 \\ . \end{cases}$$

Since $(f^{-1})^{-1} = f$, the same construction gives us a homotopy from c to $f^{-1} \cdot f$. □

The most important fundamental group to remember is $\pi_1(S^1) = \mathbb{Z}$. We will see a short proof of this later using heavier machinery, because trying to prove it directly from the definition is longer and more painful. Here is a handwave-y argument: the only nontrivial loop is the circle itself, which is a rotation by 2π. All other loops are simply concatentations of this loop by itself, namely integer multiples of the rotation.

The fundamental group (as we will see during our time with homotopy) captures several properties of the initial space, as exemplified by the following proposition.

Proposition 1.8. *Let $f : X \to Y$ be a pointed map. Define $f_\# : \pi_1(X) \to \pi_1(Y)$ by $f_\#[\gamma] = [f \circ \gamma]$[1]. Then, $f_\#$ is a group homomorphism.*

Proof. We need to show that if $\gamma \simeq \omega$, then $f_\# \circ \gamma \simeq f_\# \circ \omega$, so that $f_\#$ is well-defined. This is an easy application of Lemma 1.4 with $(W, A) = (S^1, (1,0))$. Next, to see that $f_\#$ respects the group operation, note that $f \circ (\omega \cdot \gamma) = (f \circ \omega) \cdot (f \circ \gamma)$ by an easy exercise in definition-chasing (see Exercise 2). So,

$$f_\#[\omega \cdot \gamma] = [f \circ (\omega \cdot \gamma)] = [(f \circ \omega) \cdot (f \circ \gamma)] = [f \circ \omega] \cdot [f \circ \gamma] = f_\#[\omega] \cdot f_\#[\gamma].$$

□

Remark 1.9. For the category theorist, the association $(X, x_0) \to \pi_1(X, x_0)$ with $f \to f_\#$ is a functor from the category of pointed topological spaces to the category of groups.

We have been very liberal with omitting the basepoint in our notation. To what extent does the basepoint matter to the fundamental group?

Proposition 1.10. *The group $\pi_1(Y, y_0)$ depends only on the path-connected component of Y containing y_0.*

(i) If Y_0 is the path component containing y_0, then $\pi_1(Y, y_0) \cong \pi_1(Y', y_0)$.

(ii) If y_0 and y_1 are in the same path component, then $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$.

Proof. The proof of (i) is not complicated. There is a one-to-one correspondence between pointed maps $f : S^1 \to (Y, y_0)$ and $f : S^1 \to (Y', y_0)$, the correspondence being that they are the same. Any map $f : S^1 \to Y$ must lie in one path component.

The proof of (ii) involves a little more work. Let $\alpha : [0,1] \to Y$ be a path from y_0 to y_1. Given a loop f at y_0, we can construct a loop at y_1 by travelling from y_1 to y_0 along α^{-1}, traversing f, then travelling back to y_1 along α. (See Figure 5.)

[1]Pronounced 'f-sharp'.

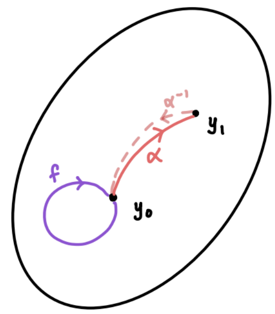


Figure 5: Creating a loop at y_1.

Define $\phi : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ by $[f] \to [\alpha \cdot f \cdot \alpha^{-1}]$, where $\alpha \cdot f \cdot \alpha^{-1}$ denotes the concatenated map $[0, 1] \to Y$. Any homotopy $f \to g$ extends naturally to a homotopy from $\alpha \cdot f \cdot \alpha^{-1}$ to $\alpha \cdot g\alpha^{-1}$, so ϕ is well-defined. (Work out the details in Exercise 3.)

ϕ is a homomorphism.

$$\phi([f][g]) = [\alpha \cdot f \cdot \alpha^{-1}][\alpha \cdot g \cdot \alpha^{-1}] = [\alpha \cdot f \cdot (\alpha^{-1} \cdot \alpha \cdot g \cdot \alpha^{-1}] = [\alpha \cdot fg \cdot \alpha^{-1}],$$

since $\alpha \cdot \alpha^{-1}$ is homotopic to the identity. (This was step 3 in the proof of Theorem 1.7.)

ϕ is injective. For any $f, g \in \pi_1(Y, y_0)$. Let c_0 and c_1 be the constant loops at y_0 and y_1 respectively. If $\phi[f] = [c_1]$, then

$$f \simeq c_0 f c_0 \simeq \alpha^{-1}\alpha \cdot f \cdot \alpha^{-1}\alpha \simeq \alpha^{-1}c_1\alpha \simeq \alpha^{-1}\alpha \simeq c_0.$$

The reader should verify that each $\simeq$ does, in fact, indicate a homotopy equivalence using the fact that $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1)$ are well-defined groups.

ϕ is surjective. For any $[g] \in \pi_1(Y, y_1)$, $f = \alpha^{-1}g\alpha$ is a loop centered at y_0, and $\phi([f]) = [g]$. □

The fundamental group is preserved by direct products.

Proposition 1.11. *Let $X = \prod_{i \in I} X_i$. For any $x = (x_i)_{i \in I} \in X$, $\pi_1(X, x) \cong \prod_{i \in I} \pi_1(X_i, x_i)$, i.e. the fundamental group is preserved by direct products.*

Proof. There is an obvious choice for the group isomorphism. Each projection $p_i : X \to X_i$ induces a group homomorphism $p_{i\#} : \pi_1(X, x) \to \pi_1(X_i, x_i)$. Let $\phi : \pi_1(X, x) \to \prod_{i \in I} \pi_1(X_i, x_i)$ be the extension $\phi([f]) = \prod_{i \in I} p_{i\#}[f]$. The map ϕ is a group homomorphism because each map $p_{i\#}$ is.

ϕ is injective. Suppose $\phi[f]$ is the identity. Then, there are homotopies $H_i : p_{i\#}[f] \to c_i$ for each $i \in I$, where c_i is (as always) the constant loop at x_i. The product homotopy $H(z, t) = \Big(H_i(z, t)\Big)_{i \in I}$ defines a homotopy from f to $(c_i)_{i \in I}$, the constant loop at $x \in X$. (See Exercise 4.) □

We will soon fall into the practice of omitting the basepoint from our notation. This poses a slight notational problem for the identity element of the fundamental group. The letter c will be reserved for the constant loop, and we will use subscripts to indicate which space we are referring to. For example, when we only consider one or two pointed spaces, say X and Y, we will call their respective loops c_X and c_Y. However, as in the previous proposition when we consider indexed spaces $(X_i)_{i \in I}$, it is convenient to refer to their loops as c_i, thereby avoiding a dreaded double subscript.

Intuitively, we would like a sphere of radius 2 to be the same as a sphere of radius 1: they have exactly the same loops, only at different scales. How do we formalize this notion?

Definition 1.12. A pointed map $f : X \to Y$ is a *homotopy equivalence* if there is a pointed map $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g 1_Y$, the identity maps on X and Y respectively. Two spaces X and Y are *homotopy equivalent* if there is a homotopy equivalence between them.

Example 1.13.

(i) Any homeomorphism between two spaces is a homotopy equivalence. This is the special case when $f \circ g$ and $g \circ f$ are equal to the identity maps.

(ii) The space $\mathbb{R}^n$ is homeomorphic to a single point, say $\{a\}$. Define $f : \mathbb{R}^n \to \{a\}$, as the constant map, and $g : \{a\} \to \mathbb{R}^n$ by $g(a) = x$, where x is our chosen basepoint. Clearly $f \circ g = 1_{\{a\}}$. Conversely, require that any two maps $\mathbb{R}^n \to \mathbb{R}^n$ are homotopic by Example 1.2, so $g \circ f \simeq 1_{\mathbb{R}^n}$.

(iii) The inclusion map from S^1 to $\mathbb{C} \setminus \{0\}$ is a homotopy equivalence. Intuitively, we expand the hole at 0 in $\mathbb{C} \setminus \{0\}$ to a circle, and compress the rest of the plane onto the circle. Formally, [2] we define a *retraction* from $\mathbb{C} \setminus \{0\}$ to S^1. Define $r : \mathbb{C} \setminus \{0\} \to S^1$ by $r(z) = \frac{z}{||z||}$. The composition $r \circ i$ is the identity on S^1. The other composition $i \circ r$ is homotopy equivalent to the identity on $\mathbb{C} \setminus \{0\}$ via the homotopy

$$H(z, t) = \frac{z}{1 + t(||z|| - 1)}.$$

Theorem 1.14. *If X and Y are homotopy equivalent, then $\pi_1(X) = \pi_1(Y)$.*

Proof. Let $f : X \to Y$ be a homotopy equivalence with inverse $g : Y \to X$. Recall that $f_\# : \pi_1(X) \to \pi_1(Y)$ is a group homomorphism. Given any loop $[\alpha] \in \pi_1(X)$, $g_\# f_\# [\alpha] = [g \circ f \circ \alpha] = [1_X \circ \alpha] = [\alpha]$. The same argument shows that $f_\# g_\#$ is the identity on $\pi_1(Y)$. We have proven a little more: every homotopy equivalence induces an isomorphism of fundamental groups. □

Exercises

Exercise 1. Let f and g be pointed maps from S^1 to Y. If $f' \simeq f$ and $g' \simeq g$ are homotopic as pointed maps, then $f \cdot g \simeq f' \cdot g'$, with the group operation as above. [3]

Exercise 2. Show that $f \circ (\omega \cdot \gamma) = (f \circ \omega) \cdot (f \circ \gamma)$ in the proof of Proposition 1.8.

Exercise 3. Let $\alpha : [0, 1] \to Y$ be a path from y_0 to y_1 and f a loop at y_0, as in the proof of Proposition 1.10. Define the map $\alpha \cdot f \cdot \alpha^{-1} : [0, 1] \to Y$ and verify that it is a loop at y_1. If $H : f \to g$ is a homotopy relative to y_0, construct a homotopy from $\alpha \cdot f \cdot \alpha^{-1} \to \alpha \cdot g \cdot \alpha^{-1}$.

Exercise 4. Show that the product homotopy $H(z, t)$ from the proof of Proposition 1.11 is in fact a homotopy from f to $(c_i)_{i \in I}$. You will need to use the construction of each $p_{i\#}$ from Proposition 1.8.

1.3 Contractibility

The last two examples in Example 1.13 showcase two important types of homotopy equivalences. Example (ii) shows *contractibility* to a point, while example (iii) is a *deformation retraction* onto a subspace. Contractibility is a special case of a deformation retraction, but noteworthy enough to merit a separate definition.

Definition 1.15. A pointed space (X, x_0) is *contractible* if $1_X \simeq c_X$, i.e. the identity map on X is homotopy equivalent to the constant map x_0.

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[2] Or, unintuitively.

[3] *Hint:* In the proof of Proposition 1.3, we concatenated two homotopies by compressing them along the second coordinate. In this case, given homotopies $F : f \to f'$ and $G : g \to g'$, concatenate them by compressing along the first coordinate.

Remark 1.16. Be careful! This is the definition of contractibility for *pointed* spaces. An arbitrary space X is contractible if the identity is homotopy equivalent to a constant map, but not necessarily through a basepoint-preserving homotopy. At the end of this section, we will see an example of a space that is (weakly) contractible to a point, but not (strongly) contractible as a pointed space with that basepoint.

Think of a contraction literally: you squeeze $\mathbb{R}^n$ to a single point. The disk D^n, for example, is contractible. Let p be the center of the disk and also its basepoint. The map $H : 1_{D^n} \to c_p$ by $H(x,t) = tp + (1-t)x$ is a homotopy.

In general, we say $f : S^1 \to X$ is *null homotopic* if it is homotopic to the constant map. A space X is then contractible if every loop is null homotopic. Of course, since homotopy is an equivalence relation, this means that *all* loops are homotopic.

Proposition 1.17. *If X is contractible and Y is any topological space, then any two maps from $Y \to X$ are homotopic. Further, X is contractible if and only if X is homotopic to a point.*

Proof. The first statement follows immediately from the definition: if $f : Y \to X$ is any map, then $f = f \circ 1_X \simeq f \circ c_X$, the constant map from Y to X. If X is contractible, then $c_X : X \to \{x_0\}$ and the embedding of the basepoint $i : \{x_0\} \to X$ are homotopy equivalences. For the easy direction, $i \circ c_X$ is the identity map on $\{x_0\}$. For the other (but still easy) direction, $c_X \circ i = c_X \simeq 1_X$ since X is contractible. Conversely, suppose X is homotopy equivalent to a point: without loss of generality, the point $\{x_0\}$. Then, there exist homotopy equivalences $f : X \to \{x_0\}$ and $g : \{x_0\} \to X$. Of course, since we only care about basepoint-preserving maps, $g : \{x_0\} \to X$ must be the embedding $i : \{x_0\} \to X$, and f can only be the constant map $c_X : X \to \{x_0\}$. The composition $c_X \circ i = c_X$ is homotopy equivalent to 1_X, which is exactly the statement that X is contractible. □

Example 1.18. Any convex subset X of $\mathbb{R}^n$ is contractible to any point $x_0 \in X$. The homotopy from 1_X to c_X is simply $H(x,t) = tx_0 + (1-t)x$. In particular, $\mathbb{R}^n$ is contractible to any point in the space.

We can use contractibility to answer an extension problem: when can we lift a map $f : S^n \to Y$ to a map $f : D^{n+1} \to Y$?

Corollary 1.19. *Let $f : S^n \to Y$ be a pointed map. Then, f extends to a map $\tilde{f} : D^{n+1} \to Y$ if and only if $f \simeq c_Y$.*

Proof. Let $i : S^n \to D^{n+1}$ be the inclusion map, and suppose f extends to a map $\tilde{f}$ on D^{n+1} so that $f = \tilde{f} \circ i$. Since D^{n+1} is contractible, $\tilde{f} \simeq c_Y$, so f is also homotopy equivalent to a constant map.

Conversely, suppose H is a homotopy from c_Y to f. Define the extension

$$\tilde{f}(x) = \begin{cases} y_0, & 0 \le ||x|| \le 1/2; \\ H\Big(\frac{x}{||x||}, 2||x|| - 1\Big), & 1/2 \le ||x|| \le 1. \end{cases}$$

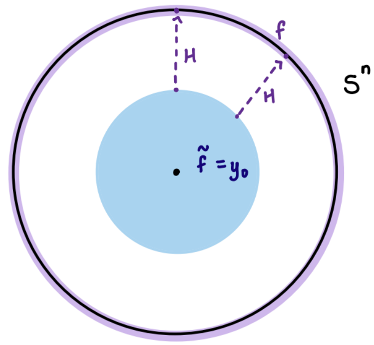


Figure 6: Extending the map f to D^{n+1}

□

We can use this result to produce a new proof of the fact that homotopy equivalence induces an isomorphism of fundamental groups.

Definition 1.20. A space Y is *simply connected* if it is path connected and $\pi_1(Y) = 1$.

Corollary 1.21. *Suppose $f, g : I \to Y$ are maps such that $f(0) = g(0)$ and $f(1) = g(1)$. If Y is simply connected, then $f \simeq g \text{rel}\{0, 1\}$.*

Proof. Since $f(0) = g(0)$ and $f(1) = g(1)$, this defines a map h on $\partial(I \times I)$, where $h(x, 0) = f(x)$, $h(x, 1) = g(x)$, $h(0, y) = f(0)$ and $h(1, y) = g(1)$.

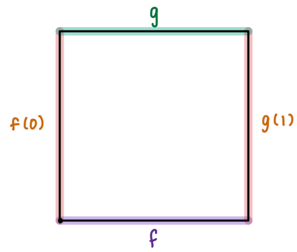
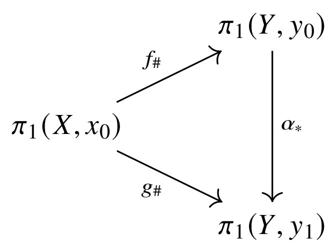


Figure 7: The map h.

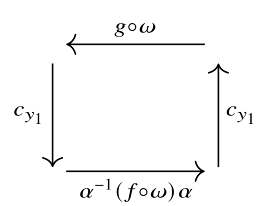
Using a homeomorphism, we can identify $I \times I$ with D^2 and $\partial(I \times I)$ with S^1. By the previous corollary, we can extend this map to S^1 if and only if the map h is null homotopic. However, since $\pi_1(Y) = 1$, h must be null homotopic. The extension of the map to S^1 and its pullback to $I \times I$ gives us the homotopy from f to g. □

Theorem 1.22. *Let f, g be maps (not necessarily pointed!) from X to Y and H a homotopy from f to g. Set $y_0 = f(x_0)$ and $y_1 = g(x_0)$, and let α be the path $\alpha(t) = H(x_0, t)$ from y_0 to y_1. Then, the following diagram commutes*



where α^* is the isomorphism that sends $[\omega] \in \pi_1(Y, y_0)$ to $[\alpha^{-1}\omega\alpha]$.

Proof. Given $[\omega] \in \pi_1(X, x_0)$, we need to show that $\alpha^* f_\#[\omega] = g_\#[\omega]$, i.e. $\alpha^{-1}(f \circ \omega)\alpha \simeq g \circ \omega$ (this time as pointed maps). We can define a map on $\partial(I \times I)$ using the same trick as the previous corollary:



Under the homeomorphism from $\partial(I \times I)$ to $S^1 \cong I/\{0, 1\}$, the oundary map becomes the concatenation $[c_{y_1}^{-1}\alpha^{-1}(f \circ \omega)\alpha c_{y_1}(g \circ \omega)^{-1}] = [\alpha^{-1}(f \circ \omega)\alpha(g \circ \omega)^{-1}]$.

$$\begin{array}{ccc} X \times I & \xrightarrow{H} & Y \\ {\scriptstyle \omega\times 1}\uparrow & & \\ S^1 \times I & & \end{array}$$

The composition in the above diagram tells us that $f \circ \omega$ is homotopic to $g \circ \omega$. □

As a corollary, if $f : X \to Y$ is a homotopy equivalence, then $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is a group isomorphism. If g is a homotopy inverse to f, let H be the homotopy from gf to 1_X and $\alpha(t) = H(x_0, t)$. The theorem tells us that the diagram commutes:

$$\begin{array}{ccc} & & \pi_1(X, x_0) \\ & \overset{(1_X)_\#}{\nearrow} & \\ \pi_1(X, x_0) & & \downarrow \alpha_* \\ & \underset{(gf)_\#}{\searrow} & \\ & & \pi_1(X, gf(x_0)) \end{array}$$

so $(gf)_\# = g_\# f_\# = \alpha_*$ is an isomorphism. By the same reasoning, $f_\# g_\#$ is an isomorphism. In this case, $f_\#$ and $g_\#$ must each be isomorphisms.

Corollary 1.23. *If X is contractible, then X is simply connected.*

Proof. Let H be a homotopy from 1_X to c_X. For any $y \in X$, define $\omega : I \to X$ by $\omega(t) = H(y, t)$. This is a path from y to x_0, so X is path connected. Since X is homotopy equivalent to a point $\{x_0\}$, $\pi_1(X, x_0) = \pi_1(\{x_0\}) = 1$. □

In general, the contractibility of a space X is not independent of the choice of basepoint $\{x_0\}$.

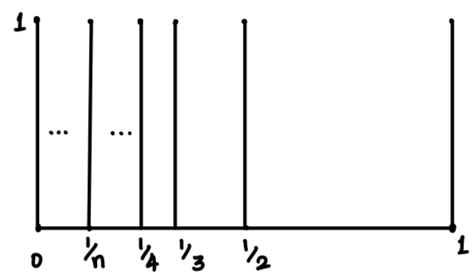


Figure 8: The comb space

Consider the comb space as a subspace of $\mathbb{R}^2$, with a vertical copy of the unit interval above each point $(1/n, 0)$ on the x-axis. More formally, the comb space is the subset X of $I \times I$, with $X = (I \times \{0\}) \cup (\{0\} \times I) \cup \Big(\cup_n (\{1/n\} \times I \Big)$.

This is a good moment to pause for a sanity check. The comb space is homotopic to a point, but does not deformation-retract onto any basepoint. (see here for a discussion of this fact). But didn't we just prove that if a space is homotopic to a point, then it deformation retracts to its basepoint? This is a problem with our definition of contractibility. In general, contractibility does not require being "relative to the basepoint", unlike deformation retractions.

2.1 THE FUNDAMENTAL GROUP OF S^1

This presentation is based on section 1.1 of Hatcher. The goal is to get our hands a little dirty computing one fundamental group. All the other fundamental groups will be computed using either Van Kampen's theorem or covering spaces.

Theorem 2.1. *The map $\phi : \mathbb{Z} \to \pi_(S^1)$ that sends n to the homotopy class of the loop $\omega_n(t) = (\cos(2\pi nt), \sin(2\pi nt))$ is an isomorphism.*

Corollary 2.2 (The fundamental theorem of algebra)**.** *Every nonconstant polynomial in $\mathbb{C}[X]$ has a root in $\mathbb{C}$.*

Proof. Suppose $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. If p has no roots in $\mathbb{C}$, then for every $r \geq 0$ and $t \in [0, 1]$,

$$f_r(t) = \frac{p(re^{2\pi it})/p(r)}{|p(re^{2\pi it})/p(r)|}$$

is well-defined. For fixed r, this defines a loop in S^1 based at 1. The loops $(f_r)_{r\geq 0}$ are homotopy equivalent (why?). The loop f_0 is the constant loop, so the class $[f_r]$ is trivial in $\pi_1(S^1)$.

Choose $r > \max(|a_0| + \cdots + |a_{n-1}|, 1)$. Then, if $|z| = r$, $|z^n| > (|a_0| + \cdots + |a_{n-1}|)|z^{n-1}| \geq |a_0 + \cdots + a_{n-1}z^{n-1}|$. It follows that for each $0 \leq s \leq 1$, the polynomial $p_s(z) = z^n + s(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0)$ has no roots on the circle $|z| = r$. We can define $f_{s,r}(t)$ analogously for the polynomials $p_s(z)$. As s goes from 1 to 0, this defines a homotopy from f_r to the loop $\omega_n(t) = e^{2\pi int}$. Since p is nonconstant, $n \geq 1$ and ω_n is a nontrivial loop in $\pi_1(S^1)$, a contradiction. □

Corollary 2.3 (Two-dimensional Brouwer fixed point theorem)**.** *Every continuous map $h : D^2 \to D^2$ has a fixed point.*

2.2 VAN KAMPEN'S THEOREM

Van Kampen's theorem is one of two powerful tools to compute homotopy groups.

Theorem 2.4 (Van Kampen, version 1)**.** *Let U and V be connected open subsets of X such that $U \cup V = X$ and $U \cap V$ is nonempty, connected, and contains the basepoint. Let $i_U : U \cap V \to U$, $i_V : U \cap V \to V$, $j_U : U \to X$, and $j_V : V \to X$ be the inclusion maps. Then,*

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V),$$

where $*$ denotes the amalgamated free product.

The amalgamated free product is essentially a noncommutative version of the direct product. In the most general case, suppose we have groups (G_α) and some group A equipped with homomorphisms $f_\alpha : A \to G_\alpha$. The *amalgamated free product with respect to A* is denoted by $G = *_\alpha G_\alpha$ and constructed in the following complicated way. The elements of G are reduced words $g_1 \cdots g_m$, where each g_i is in a different group G_α. The group operation $\circ$ is concatenation, but we are allowed to multiply letters from the same group.

That is, if $g \in G_\alpha$, then $g \circ g$ is the letter $g^2 \in G$. This is the *free product* of the groups G_α. To amalgamate it, we quotient out by the relations

$$gf_\alpha(a)h = gf_\beta(a)h.$$

if $g \in G_\alpha, h \in G_\beta$, and $a \in A$, When the homomorphisms f_α are all trivial homomorphisms (like when A is the trivial group), this is the free product of the groups G_α. For example, $\mathbb{Z} *_1 \mathbb{Z}$ is the free group on two elements.

A sequence of homomorphisms $\varphi_\alpha : G_\alpha \to H$ induces a homomorphism $\Phi : *_\alpha G_\alpha \to H$ by $\Phi(g_1 \cdots g_m) = \varphi_{\alpha_1}(g_1) \cdots \varphi(\alpha_m)(g_m)$.

Theorem 2.5 (Van Kampen, general version)**.** *Let (U_α) be a collection of connected open sets that cover X such that* *(i) each U_α contains the basepoint x_0, and*

(ii) the pairwise intersections $U_\alpha \cap U_\beta$ are nonempty and connected.

Then, the homomorphism from the (non-amalgamated!) free product $\Phi : *_\alpha \pi_1(U_\alpha) \to X$ induced by the inclusion maps is a homomorphism.

If every triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$, then $\ker(\Phi)$ is the normal subgroup generated by $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$, where $i_{\alpha\beta} : \pi_1(U_\alpha \cap U_\beta) \to \pi_1(U_\alpha)$ and $i_{\beta\alpha} : \pi_1(U_\alpha \cap U_\beta) \to \pi_1(U_\beta)$ are induced by the inclusion maps.

This is pretty complicated, and probably not typically a result you will use. Instead, a fairly standard induction argument gets you a more useful generalization.

Theorem 2.6 (Van Kampen, inductive). *Let $U_1, \cdots, U_n$ be open connected sets that cover X and each contain the basepoint. If each pairwise intersection $U_i \cap U_j$ is nonempty and connected,*

$$\pi_1(X) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) *_{\pi_1((U_1 \cup U_2) \cap U_3)} \pi_1(U_3) \cdots *_{(\pi_1(U_1) \cup \cdots \pi_1(U_{n-1})) \cap U_n)} \pi_1(U_n).$$

2.3 Applications of Van Kampen

Example 2.7 (Fundamental group of a wedge sum.). Let (X_α, x_α) be a collection of spaces such that each x_α is the deformation retract of an open neighborhood V_α in X_α. Let $U_\alpha = X_\alpha \bigvee_{\beta \neq \alpha} V_\beta$. The sets U_α cover $\bigvee X_\alpha$, and the pairwise and triple intersections deformation retract to the basepoint. Further, the spaces $U_\alpha \cap U_\beta$ are contractible and have trivial fundamental group, so the homomorphisms $i_{\alpha\beta}$ are all trivial. Van Kampen's theorem now implies that

$$*_\alpha \pi_1(X_\alpha) \cong \pi_1(\bigvee_\alpha X_\alpha).$$

Example 2.8 (Fundamental group of a graph.). Let G be a connected graph with n vertices and m edges. I claim that $\pi_1(G)$ is the free group on $m - n + 1$ elements.

Let T be a spanning tree of G, so T has $(n-1)$ vertices. For each edge $e_i \notin T$, let U_i be an open neighborhood of T that contains e_i (but no other extra edges). Then, U_i deformation retracts to T. The connected open sets U_i cover G and their pairwise intersections deformation retract to T, which is contractible. So, Van Kampen's theorem tells us

$$\pi_1(X) = *_{e_i \notin T} \mathbb{Z},$$

which is the free group generated by $m - n + 1$ elements.

3 An introduction to homology

While the homotopy groups are easy to define, they are typically difficult to compute. Even the higher homotopy groups of n-spheres are notoriously difficult to compute. The *homology groups*, on the other hand, are much easier to compute because they usually terminate in a chain of zeroes. They also have the added advantage of being abelian.

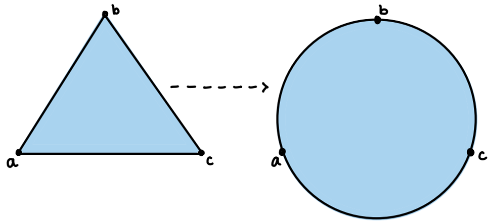


Figure 9: The triangulation of D^2

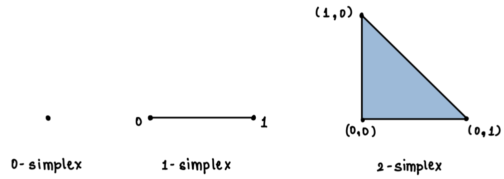


Figure 10: The 0–, 1–, and 2–simplices

.

The nth homotopy group tells you about n-dimensional holes in your space by looking at homotopy classes of loops. The homology groups do something similar: they identify holes by looking at *triangulations* of the space. For example, we can think of the disk D^2 as a (filled-in) triangle, which is contractible, so D^2 has no holes in any dimension. (They are homeomorphic, after all.) Of course, once we start looking at higher-dimensional objects, like D^3, we have to use uncountably many two-dimensional triangles to cover the space, and this becomes meaningless. This motivates our use of the *n-simplex*:

Definition 3.1. The *standard n-simplex*, denoted Δ^n, is the convex hull of the standard basis vectors in $\mathbb{R}^n$ and the origin with a linear order on its vertices determined by the coordinates.

In general, an n-simplex in $\mathbb{R}^n$ is a convex hull of $n + 1$ affinely independent vectors in $\mathbb{R}^n$ equipped with a linear order. While we do not need to use the linear order for CW-complexes (yet), we will need it to define Δ-complexes, and to construct the homology groups. Some facts about n-simplices:

(1) The convex hull of $m + 1$ vertices, where $m \leq n$, is called an *m-face* of the n-simplex. Every m-face is an m-simplex.

(2) The boundary $\partial\Delta^n$ is the union of the $(n - 1)$-faces.

(3) The m-faces are in one-to-one correspondence with the $(m + 1)$-sets of vertices.

(4) A 0-face is a vertex and a 1-face is an edge.

Now, for example, we can triangulate D^3 via a homeomorphism to a tetrahedron. In general, the disk D^n is homeomorphic to the simplex Δ^n. A triangulation, loosely speaking, is a collection of maps from n-simplices to our space. For example, here are two triangulations of S^1. One identifies the vertices of Δ^1 (the unit interval $[0, 1]$) with a single point Δ^0. The other is simply the homemorphism from S^1 to the boundary of $\partial(\Delta^2)$, which consists of three 1-simplices (edges) and three 0-simplices (vertices).

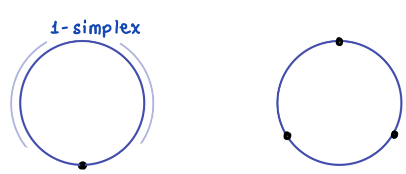


Figure 11: Two triangulations of S^1

3.1 Complex? I find it quite simple, really

Since a collection of maps from simplices to a space X produces a more complicated structure, we call X a *complex*, where the type of complex depends on the maps. The three types we will look at are

$$\text{simplicial complex} \subseteq \Delta\text{-complex} \subseteq \text{CW-complex}.$$

We will repeatedly switch between thinking of the n-simplex as the standard n-simplex, and as the homemorphic space D^n, so that the boundary of the n-simplex is homeomorphic to S^{n-1}.

3.1.1 CW–complexes

The easiest definition of a CW-complex is the inductive construction.

Definition 3.2. A CW-complex is a space X formed in the following way.

1. Begin with a discrete space X^0. (These are the points of X equipped with the discrete topology.)
2. Suppose we have constructed X^{n-1}. Take a collection of maps $f_\alpha : S^{n-1} \to X^{n-1}$, for α in some index set J_n. The space X^n is formed by taking the union of X^{n-1} with J_n copies of D^n,[4] and taking the quotient by the equivalence relation $x \sim f_\alpha(x)$. More precisely,
$$X^n = \left(X^{n-1} \cup \bigcup_\alpha D^n\right) \big/ \sim.$$
3. If we stop the construction after finitely many steps, $X = X^n$ for some n. Otherwise, we inductively perform this construction for every $n \in \mathbb{N}$, and $X = \cup_n X^n$. In the latter case, we equip X with a weak topology: a set $A \subset X$ is closed if and only if $A \cap X^n$ is closed for every $n \in \mathbb{N}$.

Step two is the heart of the construction; the maps f_α glue the boundaries of the n-simplices to the $(n-1)$-simplices in X^{n-1}. Since the only restriction we have is that the f_α's be continuous, this gluing does not have to be nice.

Definition 3.3. The interior of the disk D^n (or the simplex Δ^n) is called an *n-cell*. When $n = 0$, a 0-simplex is a 0-cell.

This is useful terminology because when we glue Δ^n to X^{n-1}, we identify $\partial(\Delta^n)$ with simplices in X^{n-1}, and what remains is the n-cell, $\text{int}(\Delta^n)$.

Example 3.4.

(i) The sphere S^n is formed by first taking a 0-cell, e_0, and attaching Δ^n to it by identifying the boundary of Δ^n with the single point e_0.

(ii) We can also form the sphere S^n with a finer structure by identifying it with the boundary $\partial(\Delta^n)$.

(iii) Here is a non-example. We can write the sphere S^n as a disjoint union of 0-cells, $S^n = \cup_{x \in S^n}\{x\}$. This is not a CW-complex, for the silly reason that it violates the first condition: S^n is not a discrete space. More generally, this is also why we have the third condition to prevent trivial complexes that are just the points of the space.

An equivalent way to define a CW-complex is by defining the gluing maps to act on all of D^n, not just its boundary.

Definition 3.5. A CW-complex is a space X that can be formed in the following way.

1. Begin with a discrete space X^0. (These are the points of X equipped with the discrete topology.)
2. Suppose we have constructed X^{n-1}. We have a collection of maps $\sigma_\alpha : \Delta^n \to X$, for α in some index set J_n. Each σ_α is injective on the interior $\text{int}(\Delta^n)$, and the restriction to the boundary is a map $\sigma_\alpha : \partial(\Delta^n) \to X^{n-1}$. Define X^n to be the union of X^{n-1} with J_n copies of Δ^n, quotiented by the equivalence relation $x \sim f_\alpha(x)$. More precisely,
$$X^n = \left(X^{n-1} \cup \bigcup_\alpha D^n\right) \big/ \sim.$$
3. Either $X = X^n$ for some n, or $X = \cup_{n=0}^\infty X^n$. In the latter case, we equip X with a weak topology: a set $A \subset X$ is closed if and only if $A \cap X^n$ is closed for every $n \in \mathbb{N}$.

In step 2, the injectivity of the embedding on the interior of Δ^n is exactly the same as only gluing the boundary of Δ^n to X^{n-1}, so the interior is preserved.

[4] Remember, we identify D^n with Δ^n!

3.1.2 Δ-complexes

The construction of S^n with one 0-cell and one n-cell is not very natural for $n \geq 2$. We would like to glue the boundary of an n-simplex to $(n-1)$-simplices. A Δ-complex is a CW complex where the restriction of the gluing maps $\sigma_\alpha : \Delta^n \to X$ to each $(n-1)$-face of Δ^n is an inclusion map.

Definition 3.6. A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ (where α depends on n) such that

(i) Each $\sigma_\alpha : \Delta^n \to X$ is injective on the interior of Δ^n, and each $x \in X$ is in the image of exactly one such interior.

(ii) The restriction of $\sigma_\alpha : \Delta^n \to X$ to every $(n-1)$-face of Δ^n is another map $\sigma_\beta : \Delta^{n-1} \to X$. We identify the face with Δ^{n-1} with using the linear homemorphism that preserves the order of the vertices.

(iii) A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for every map.

Again, the last condition is to prevent silly structures like considering X as a collection of 0-cells. An important consequence of the Δ-complex structure is that if we have a map $\sigma_\alpha : \Delta^n \to X$ for some n, then the Δ-complex must contain maps $\sigma_\beta : \Delta^k \to X$ for every $k < n$ (by restricting σ_α to the faces of Δ^n). The CW-complex structure of S^n as an n-cell and a 0-cell is not a Δ-complex for $n \geq 2$.

Example 3.7.

1. The homeomorphism from $\partial(\Delta^n) \to S^{n-1}$ is a Δ-complex structure.
2. Here is a Δ-complex structure on the torus.

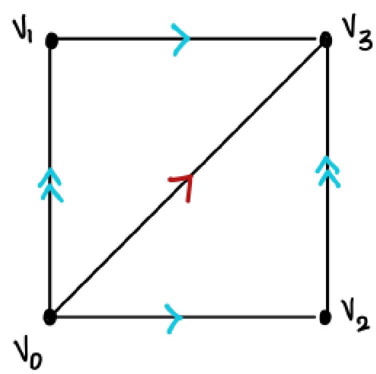


Figure 12: A Δ-complex structure on the torus

3. Here is a structure that is not a Δ-complex structure on the torus

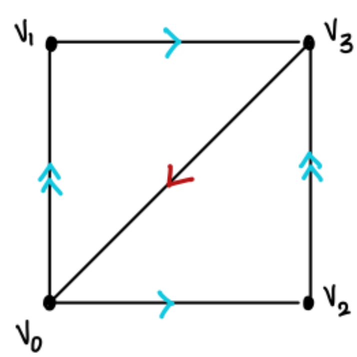


Figure 13: Not a Δ-complex structure on the torus

3.1.3 Simplicial complexes

Simplicial complexes have even more structure than Δ-complexes.

Definition 3.8. A simplicial complex structure on a space X is a Δ-complex structure such that

(i) Each $\sigma_\alpha : \Delta^n \to X$ maps different faces of Δ^n to different $(n-1)$-simplices in X^{n-1}.

(ii) The image of each simplex in X is uniquely determined by its vertices.

For the combinatorialists, here is another definition.

Definition 3.9. A simplicial complex $\mathcal{K}$ is a collection of simplices such that

(i) Every face of a simplex in $\mathcal{K}$ is also in $\mathcal{K}$, and

(ii) the nonempty intersection of any two simplices $\sigma_1, \sigma_2 \in \mathcal{K}$ is a face of both σ_1 and σ_2.

A simplicial complex structure on a space X is a homeomorphism from a simplicial complex to X.

Example 3.10.

1. The Δ-complex structure on S^1 formed by attaching a 1-cell to a 0-cell is not a simplicial complex. Let $\sigma_0 : \Delta^0 \to X$ and $\sigma_1 : \Delta^1 \to X$ be the attaching maps. The map $\sigma_1 : \Delta^1 \to S^1$ maps both vertices of Δ^1 to the same point in X, violating (i). The structure also violates (ii): the vertex Δ^0 determines both maps σ_0 and σ_1.
2. The homeomorphism $\partial(\Delta^n) \to S^{n-1}$ is a simplicial complex structure.

Any Δ-complex structure can be subdivided to create a simplicial complex structure. Here is an example for the sphere S^1 with the Δ-complex structure of a 1-cell and a 0-cell.

Figure 14: By subdiving S^1 enough times, we create a simplicial complex structure.

Of course, the structure we have created is just the homeomorphism $\partial(\Delta^2) \to S^1$, but the method is instructive.

3.2 The singular homology groups

Now we will forget all about our complexes and construct the most general homology groups: the *singular ones*. Instead of considering maps $\sigma : \Delta^n \to X$ that have a nice structure, we will consider *all* possible continuous maps $\sigma : \Delta^n \to X$. This has the advantage of being general, but the disadvantage of being computationally unwieldy. We will later see that we can define homology groups using the nicer structures from the previous subsection, and that these homology groups are exactly the same.

Definition 3.11. For every $n \in \mathbb{N}$, define the *chain group* $C_n(X)$ as the free abelian group generated by all possible maps $\sigma : \Delta^n \to X$. That is,

$$C_n(X) = \{\sum_\alpha n_\alpha \sigma_\alpha : \sigma_\alpha : \Delta^n \to X\},$$

where we only allow finite sums. The elements of $C_n(X)$ are called *n-chains* of X.

The formal sums in $C_n(X)$ are similar to the concatenation of loops we saw in the fundamental group. Since the simplices are oriented, we can think of its image under a map σ as oriented in X. The element $n \cdot \sigma$ involves walking along the image of σ n times, where the direction of the walk depends on whether n is positive. This analogy does not extend well to formal sums $\sum n_\alpha \sigma_\alpha$ where the images of each σ_α might be disjoint. To force an extension, we can imagine a person walking along σ_α n_α times, and then jumping to the next map.

Remember, the intuition for homology is to measure the holes in the space. So, we need to look at the boundaries of the maps σ_α.

Figure 15: The orientation-preserving walk along the boundary of Δ^2 uses the edges $[v_0v_1], [v_1v_2], -[v_0v_2]$.

Definition 3.12. Define the *boundary map* $\partial_n : C_n(X) \to C_{n-1(X)}$ as follows. Write the n-simplex Δ^n as $[v_0, \cdots, v_n]$, where the vertices are ordered. Let $[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ be the $(n-1)$-simplex with vertices $\{v_j : j \neq i\}$. This is the unique $(n-1)$-face of Δ^n that does not contain the vertex v_i. Given a map $\sigma : \Delta^n \to X$, let $\sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n]$ denote the restriction of σ to the $(n-1)$-face. Define

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_n] \in C_{n-1}(X)$$

and extend it linearly to the free abelian group $C_n(X)$. When $n = 0$, $\delta_0 : C_0(X) \to 0$ is the zero map.

The reason for the factor of $(-1)^i$ is to make sure that our imaginary walk along Δ^n is orientation-preserving. Consider the following example with the triangle Δ^2:

Lemma 3.13. *The boundary maps satisfy $\delta_n \circ \delta_{n+1} = 0$.*

Proof. Given $\sigma : \Delta^{n+1} \to X$,

$$\begin{aligned}
\delta_n\delta_{n+1}(\sigma) &= \delta_n \sum_{i=0}^{n+1} (-1)^i \sigma|[v_0, \cdots, \hat{v}_i, \cdots, v_{n+1}] \\
&= \sum_{i=0}^{n+1} \sum_{j<i} (-1)^{j+i} \sigma|[v_0, \cdots, \hat{v}_i, \hat{v}_j, \cdots, v_{n+1}] + \sum_{i=0}^{n+1} \sum_{j>i} (-1)^{j+1+i} \sigma|[v_0, \cdots, \hat{v}_i, \hat{v}_j, \cdots, v_{n+1}] \\
&= 0.
\end{aligned}$$

Each term $[v_0, \cdots, \hat{v}_i, \cdots, v_{n+1}]$ is an $(n-1)$-simplex in its own right. When we remove vertex j from the order, the parity of its appearance depends on whether $j < i$ or $j > i$. This is why the second boundary map splits into two sums that cancel each other.

Extending this linearly yields the result that $\delta_n \circ \delta_{n+1} = 0$ on the entire chain group.[5] □

Now we are ready to define the homology groups. The previous lemma tells us that the sequence $\left(C_n(X)\right)_{n\in\mathbb{N}}$ forms what is called a *chain complex* with the boundary maps.

Definition 3.14. A chain complex is a sequence of groups $(C_n)_{n\in\mathbb{Z}}$ with connecting homomorphisms $\delta_n : C_n \to C_{n-1}$ such that $\delta_n \circ \delta_{n+1} = 0$.

Homology groups can be defined for any chain complex, but for now we will focus on our topological motivations. For our homology groups, we only consider chain complexes where $C_n = 0$ for $n < 0$.

Definition 3.15. The nth *singular* homology group of X is defined as

$$H_n(X) = {}^{\ker \delta_n}\!/\!_{\mathrm{im}\delta_{n+1}}.$$

The elements of $\mathrm{im}\delta_{n+1}$ are called *boundaries* because they arise as the boundaries of $(n+1)$-chains. The elements of $\ker \delta_n$ are called *cycles* because their boundaries "cycle around" to cancel out.

For an arbitrary space X, the simplest singular chain group we can consider is $C_0(X)$. A continuous map from a 0-simplex to X is just a map from a 0-simplex to a point of X. So, the chain group $C_0(X)$ is the free abelian group generated by the points of X. Since δ_0 is the zero map, $\ker \delta_0 = C_0(X)$. The image of a 1-chain under the boundary map δ_1 is of the form $x - y$, where $x, y \in X$. A 1-chain is just a path in X, so $x - y$ is the image of a 1-chain if and only if x and y are in the same path-connected component of X. So, $\mathrm{im}\delta_1$ is the free abelian group generated by $\{x - y : x \text{ and } y \text{ are in the same path component}\}$. This proves the following proposition:

[5] Don't forget to extend to the chain group!

Proposition 3.16. $H_0(X) \cong \bigoplus_{i \in I} \mathbb{Z}$*, where* I *indexes the path components of* X*. In particular,* $H_0(X) \cong \mathbb{Z}$ *if and only if* X *is path-connected.*

We already have a topological characterization of X using a homology group!

Next, the group $H_1(X)$ bears a striking resemblance to $\pi_1(X)$. A map $\sigma : \Delta^1 \to X$ is in the kernel of δ_1 exactly when its endpoints are the same: it is a loop. As we know, a loop is just a continuous map $S^1 \to X$. However, the chain group $C_1(X)$ and its homology group $H_1(X)$ are always abelian, while $\pi_1(X)$ need not be. This is because $H_1(X)$ is the abelianization of $\pi_1(X)$: $H_1(X) \cong {}^{\pi_1(X)}/_{\pi_1'(X)}$, where $\pi_1'(X)$ is the commutator subgroup. It is not too difficult to show that there is a natural quotient map $\pi_1(X) \to H_1(X)$. The trick is to show that the kernel of this map is exactly the commutator subgroup, so we will postpone this proof for later.

Example 3.17 (The singular homologies of a point.). If X is a point, then

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0; \\ 0, & n > 0. \end{cases}$$

The only continuous map from an n-simplex to X is the constant map σ_n. So, the chain group $C_n(X)$ is the infinite cyclic group generated by σ_n. The boundary map sends σ_n to $\sum_{i=0}^{n}(-1)^i \sigma_{n-1}$. This image is equal to 0 when n is odd and to σ_{n-1} when n is even. So, the boundary maps δ_n are alternatingly zero and isomorphisms, except at $C_0(X)$.

$$\cdots \xrightarrow[\delta_{n+1}]{0} \mathbb{Z} \xrightarrow[\delta_n]{\cong} \mathbb{Z} \xrightarrow[\delta_{n-1}]{0} \cdots \xrightarrow[\delta_1]{0} \mathbb{Z} \xrightarrow[\delta_0]{0} 0.$$

The only nonzero homology group of this complex is $H_0(X) = \mathbb{Z}$.

Exercise 5. Prove Proposition 3.16. [6]

3.3 Other homology groups

The higher singular homology groups of arbitrary are not so easy to compute. There are far too many continuous maps from $\Delta^n \to X$ for us to realistically work with. This is why we return to the various complex structures from the previous subsection to make our chain groups smaller.

Suppose X is a CW-complex. From Definition 3.5, we have a collection of maps $\sigma_\alpha : \Delta^n \to X$ for every $n \geq 0$. Each σ_α is injective on the interior of Δ^n, so we denote the image of $\sigma_\alpha|\int(\Delta^n)$ as an n-cell e_n^α. The nth cellular chain group is $C_n^{CW}(X)$, the free abelian group generated by the n-cells e_n^α. The boundary maps δ_n have a much more complicated definition that requires some machinery. We will work through it in **??**. Assuming, for now, that the boundary maps can be defined to give $C_n^{CW}(X)$ a chain complex structure, we can define the *cellular homology groups* $H_n^{CW}(X)$.

Example 3.18 (The cellular homology of the spheres.). The sphere S^n has the CW-complex structure of an n-cell e_n attached to a 0-cell e_0. The only nonzero chain groups are $C_n^{CW}(S^n) = \mathbb{Z}[e_n]$ and $C_0^{CW}(S^n) = \mathbb{Z}[e_0]$.

$$\cdots \xrightarrow{\delta_{n+1}} \mathbb{Z} \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{\delta_0} 0$$

For $n \geq 2$, this immediately tells us that the only nonzero cellular homology groups are $H_n^{CW}(S^n) = \mathbb{Z}$ and $H_0^{CW}(S^n) = \mathbb{Z}$. For $n = 1$, we have to pay closer attention to the map $\delta_1 : C_1^{CW}(S^1) \to C_0^{CW}(S^1)$. Waving my hands, the boundary of the 1-cell e_1 is just the vertex e_0, so $\delta_1(e_1) = e_0 - e_0 = 0$. This shows that δ_1 is the identically zero map and $H_1^{CW}(S^1) = H_0^{CW}(S^1) = \mathbb{Z}$. Of course, I have not told you what δ_1 is for the cellular chain complex, so this is not a very satisfying (or even good) argument.

[6] Hint: what is ${}^{\ker \delta_0}/_{\mathrm{im}\,\delta_1}$ when X is path-connected?

The trouble with defining the cellular boundary map is that, as in the case of the spheres, there may be n-cells but no $(n-1)$-cells. This problem is easily resolved by looking at Δ-complexes, which require that the restriction of any map $\sigma : \Delta^n \to X$ to an $(n-1)$-face must be another map.

Suppose X is a Δ-complex. For each $n \in \mathbb{N}$, we have a collection of maps $\{\sigma_\alpha : \Delta^n \to X : \alpha \in J_n\}$. Let $C_n^\Delta(X)$ be the free abelian group generated by the maps σ_α. Define the boundary map $\delta_n : C_n^\Delta(X) \to C_{n-1}^\Delta(X)$ by

$$\delta_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \cdots, \hat{v_i}, \cdots, v_n].$$

This is well-defined because X is a Δ-complex, and the same proof of Lemma 3.13 shows that the groups $\left(C_n^\Delta(X)\right)$ form a chain complex. We call these the *semi-simplicial chain groups*, and the corresponding homology groups $H_n^\Delta(X)$ the semi-simplicial homology groups.

Example 3.19 (The semi-simplicial homology of the torus.).

Finally, the *simplicial homology groups* are defined in exactly the same way. Given a simplicial complex structure on X, $C_n^S(X)$ is the free abelian group generated by the n-simplices. The boundary maps are defined in exactly the same way, as are the homology groups. Since every simplicial complex is also a Δ-complex, the simplicial homology groups are the same as the semi-simplicial homology groups when we consider the structure as a Δ-complex instead.

Remark 3.20 (A very important remark about notation.)**.** Several authors prefer to denote C_n^Δ and C_n^S by Δ_n and S_n respectively. I am of the opinion that this makes it easy to forget that they are chain groups, so I will use the more cumbersome superscript notation.

Another reason for preferring this notation is that regardless of which chain groups you start out with, the resulting homology groups are all the same. There are two nontrivial results hiding in that statement. The first is that, for example, the cellular homology groups are *well-defined*. Given any two CW-complex structures on X, even if the chain groups differ, the homology groups are isomorphic. This is true for semi-simplicial and simplicial homologies as well. The second statement is that the singular, cellular, semi-simplicial, and simplicial homology groups are all the same. We will see a proof of this later.

Exercise 6. Let X and Y be topological spaces. Show that the singular homology groups are $H_n(X \sqcup Y) = H_n(X) \oplus H_n(Y)$ for all $n \in \mathbb{N}$. Generalize this to arbitrary disjoint unions $\sqcup_\alpha X_\alpha$.

4 Computing homology groups

There are two ways to compute the homology groups of a space X. The first is to directly use a complex structure (like a Δ-complex) to derive explicit expressions for the chain groups and boundary maps. This has the advantage of being a direct and elementary method but, as we saw with Example 3.19, can quickly become an involved process. With this approach, we are also freely using the fact that all types of homology groups are isomorphic.

The second approach to computation is what I will call the long exact sequence approach. First, we decompose your space X into spaces whose homology groups we know. We then use a long exact sequence of homology groups where the only unknown groups are the homology groups of X, and use exactness to recover all the groups.

There are two important long exact sequences for this purpose. The first is the long exact sequence of *relative* homologies. Using the *Excision theorem* (subsection 4.3), this tells us the homology groups of the quotient spaces X/A for good pairs (X, A). The second is the *Mayer-Vietoris* sequence, which is the homological analog of Van Kampen's theorem.

4.1 Maps between chain complexes

Before we get into computations, we need to do some book-keeping. For example, we need to show that homotopy equivalent spaces have the same homology groups. There are two purely algebraic results that will lay the foundation for the rest of this section. Recall the definition of an arbitrary chain complex from Definition 3.14.

Definition 4.1. Let (A_n, α_n) and (B_n, β_n) be chain complexes. A *map between chain complexes* is a sequence of homomorphisms $f_n : A_n \to B_n$ such that the following diagram commutes.

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\alpha_{n+1}} & A_n & \xrightarrow{\alpha_n} & A_{n-1} & \xrightarrow{\alpha_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow[\beta_{n+1}]{} & B_n & \xrightarrow[\beta_n]{} & B_{n-1} & \xrightarrow[\beta_{n-1}]{} & \cdots
\end{array}$$

In other words, $\beta_n \circ f_n = f_{n-1} \circ \alpha_n$.

4.1.1 Induced homomorphisms

Proposition 4.2. *A map between chain complexes induces homomorphisms $f_* : H_n(A) \to H_n(B)$ between homology groups.*

Remark 4.3 (An apology for notation.)**.** The astute reader will notice some sloppy notation in the previous proposition. The maps f_* should depend on n. The notation $H_n(A)$ has not been defined, even if it is clear from context. It is common for algebraic topologists (and perhaps, algebraists in general) to denote a chain complex by $A_\bullet$. This means that the nth chain group is A_n. Similarly, $H_*(A)$ denotes the sequence of homology groups of the chain complex $A_\bullet$, where the nth homology group is $H_n(A)$.

As for the functions f_*, a double subscript is cumbersome and the index n will usually be clear from context. When it is not clear, I will use a double subscript. We will avoid relegating any indices to the superscript as that is reserved for the cohomology groups.

Proof of Proposition 4.2. We want to define the induced map $f_* : H_n(A) \to H_n(B)$ by $f_*[a] = [f_*(a)]$, where the square brackets denote the homology class of an element. We only need to check that this is well-defined: f_n sends cycles to cycles and boundaries to boundaries. If $\alpha_n(a) = 0$ for some $a \in A_n$, then $\beta_n f_n(a) = f_{n-1}\alpha_n(a) = 0$. Similarly, if $a \in \text{im}\alpha_{n+1}$, so $a = \alpha_{n+1}(a')$ for some $a' \in A_{n+1}$, then $f_n(a) = \beta_{n+1} f_{n+1}(a') \in \text{im}\beta_{n+1}$. □

From now on, we will simply say $f : A_\bullet \to B_\bullet$ is a map between chain complexes, where f signifies the sequence of maps $(f_n)_{n\in\mathbb{N}}$.

Corollary 4.4.

(i) If $f : A_\bullet \to B_\bullet$ and $g : B_\bullet \to C_\bullet$ are maps between chain complexes, then $(gf)_* = g_* f_*$. That is, the induced maps agree on each homology group $H_n(A)$.

(ii) If $1 : A_\bullet \to A_\bullet$ is the identity map, then 1_* is the identity map on the homology groups.

Given a map $f : X \to Y$, we can define a map $f : C_n(X) \to C_n(Y)$ by composition: if $\sigma : \Delta^n \to X$, then $f \circ \sigma : \Delta^n \to Y$. This induces a map between the chain complexes $C_\bullet(X)$ and $C_\bullet(Y)$.

Corollary 4.5. *If two maps $f, g : X \to Y$ are homotopic, then the induced maps f_* and g_* on the homologies are the same. In particular, if X and Y are homotopy equivalent spaces, they have the same homology groups.*

Unfortunately, it is not true that if two spaces have the same homology groups, then they are homotopy equivalent. However, as a very important corollary.

Corollary 4.6. *If X is contractible, then $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$ for $n > 0$. In particular, $H_0(\Delta^k) = \mathbb{Z}$ and $H_n(\Delta^k) = 0$ for $n > 0$.*

4.1.2 Short exact sequences

In order to derive our two long exact sequences for homology, we need to start with short exact sequences of chain complexes. See **??** for background on exact sequences.

Definition 4.7. We say $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$ is a *short exact sequence of chain complexes* if $0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \to 0$ is a short exact sequence of chain complexes that commutes with the boundary maps.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\alpha_{n+1}} & A_n & \xrightarrow{\alpha_n} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle i_{n+1}} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n-1}} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{\beta_{n+1}} & B_n & \xrightarrow{\beta_n} & B_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle j_{n+1}} & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\gamma_{n+1}} & C_n & \xrightarrow{\gamma_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 16: Each column is a short exact sequence and the diagram commutes.

Theorem 4.8. *A short exact sequence of chain complexes* $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$ *induces a long exact sequence of homology groups.*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\delta_*} H_{n-1}(A) \to \cdots$$

Proof. The maps i_* and j_* are just the maps on homology induced by the maps i and j on the chain complexes. The most important thing to understand is the definition of the map $\delta_* : H_n(C) \to H_{n-1}(A)$. Let $c \in \ker \gamma_n$. Since j_n is surjective, there is some $b \in B_n$ such that $j_n(b) = c$. The element $\beta_n(b) \in \ker(j_{n-1})$, since $j_{n-1}\beta_n(b) = \gamma_n j_n(b) = 0$ by definition. We know that $\mathrm{im} i_{n-1} = \ker j_{n-1}$, so there is some $a \in A_{n-1}$ such that $i_{n-1}(a) = \beta_n(b)$. Define $\delta_*([c]) = [a]$.

$$
\begin{array}{ccc}
 & & A_{n-1} \\
 & & \downarrow{\scriptstyle i_{n-1}} \\
B_n & \xrightarrow{\beta_n} & B_{n-1} \\
\downarrow{\scriptstyle j_n} & & \\
C_n & &
\end{array}
$$

I refer the reader to Hatcher, p. 116–117 to verify that this map is well-defined, and to check that the sequence is long exact. □

4.1.3 Naturality

There is some categorical definition for naturality here that involves natural transformations between functors. Maybe I will formalize it someday. Here is all you need to know:

Theorem 4.9. *Let* $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$ *and* $0 \to X_\bullet \xrightarrow{k} Y_\bullet \xrightarrow{l} Z_\bullet \to 0$ *be a short exact sequence of chain complexes. Suppose there are chain maps* $f : A_\bullet \to X_\bullet$, $g : B_\bullet \to Y_\bullet$, *and* $h : C_\bullet \to Z_\bullet$ *such that the following diagram commutes.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & C_n & \longrightarrow & 0 \\ \downarrow & & \downarrow f_n & & \downarrow g_n & & \downarrow h_n & & \downarrow \\ 0 & \longrightarrow & X_n & \xrightarrow[k_n]{} & Y_n & \xrightarrow[l_n]{} & Z_n & \longrightarrow & 0 \end{array}$$

Then the induced maps on the long exact sequence of homologies form a commutative diagram.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{j_*} & H_n(C) & \xrightarrow{\delta_*} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow f_* & & \downarrow g_* & & \downarrow h_* & & \downarrow f_* & & \\ \cdots & \longrightarrow & H_n(X) & \xrightarrow[k_*]{} & H_n(Y) & \xrightarrow[l_*]{} & H_n(Z) & \xrightarrow[\delta_*]{} & H_{n-1}(X) & \longrightarrow & \cdots \end{array}$$

This is a very wordy result with a lot of moving parts. We will see how this comes in handy once we have set up our long exact sequences in the rest of this section.

Exercise 7. Prove Corollary 4.4.

4.2 Reduced and relative homologies

4.2.1 Reduced homologies

Sometimes, it is more convenient to consider a homology theory where all the homologies of a point are zero, Once we see an axiomatic approach to homology, the reason for preferring reduced homology will be clear: the suspension axiom holds in all degrees. For now, the reduced homology groups will give us nicer expressions for the homology groups of product spaces and wedge sums.

Definition 4.10. Define the *augmented* (singular) chain complex as

$$\cdots \to C_1(X) \xrightarrow{\delta_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0,$$

where $\epsilon\Big(\sum_i n_i\sigma_i \Big) = \sum_i n_i$. The *reduced homology groups* $\tilde{H}_n(X)$ are the homology groups of this chain complex.

For $n > 0$, the reduced homology groups are the same as the homology groups. For $n = 0$, $\tilde{H}_0(X) = \ker \epsilon$ and $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. As a result, whenever X is path-connected, $\tilde{H}_0(X) = 0$. This is also more in line with our intuition that a path-connected space should have no 0-dimensional holes.

We will use reduced homologies in long exact sequences of homologies. A short exact sequence of chain complexes induces a short exact sequence of augmented chain complexes.

4.2.2 Relative homologies

The motivation for relative homology is to understand the homology groups of the quotient space X/A, where A is a subspace of X. It is not always true that $H_n\big(X/A\big) = H_n(X)/H_n(A)$. In fact, it does not always make sense to describe $H_n(A)$ as a subgroup of $H_n(X)$. Instead, the *relative homology groups* $H_n(X, A)$ measure the difference between $H_n(X)$ and $H_n(A)$. For what Hatcher calls good pairs (X, A), these end up being the homology groups of the quotient space.

Any n-chain $\sigma : \Delta^n \to A$ extends to an n-chain $i \circ \sigma : \Delta^n \to X$ via the inclusion map. This embeds the group $C_n(A)$ as a subgroup of $C_n(X)$. The restriction of the boundary map $\delta_n^X : C_n(X) \to C_{n-1}(X)$ to the subgroup $C_n(A)$ restricts to a map $C_n(A) \to C_{n-1}(A)$. This tells us that the groups $\Big(C_n(X)/C_n(A), \delta_n\Big)$ form a chain complex.

Definition 4.11. The *relative homology groups* of the pair (X, A) are the homology groups $H_n(X, A)$ of the chain complex $\Big(C_n(X)/C_n(A), \delta_n\Big)$.

Proposition 4.12. *When $A = \{x_0\}$ is a point, the relative homology groups $H_n(X, A)$ are the same as the reduced homology groups $\tilde{H}_n(X)$.*

Proof. Let δ_n^{rel} be the boundary maps of the relative chain complex, and let δ_n be the usual boundary map for $C_n(X)$ and $C_n(A)$.

When n is odd, δ_n is identically zero on $C_n(A)$, and when n is even it is an isomorphism. So, when n is odd,

$$\begin{aligned}\ker \delta_n^{rel} &= {\ker \delta_n}/{C_n(A)}, \text{ since } C_n(A) \subset \ker(\delta_n) \\ \mathrm{im}\delta_n^{rel} &= {\mathrm{im}\delta_n + C_{n-1}(A)}/{C_{n-1}(A)}, \text{ and } \mathrm{im}\delta_n \cap C_{n-1}(A) = \{0\}.\end{aligned}$$

Similarly, when n is even and $n > 0$,

$$\begin{aligned}\ker \delta_n^{rel} &= {\ker \delta_n + C_n(A)}/{C_n(A)}, \text{ and } \ker \delta_n \cap C_n(A) = \{0\} \\ \mathrm{im}\delta_n^{rel} &= {\mathrm{im}\delta_n}/{C_{n-1}(A)}, \text{ since } C_{n-1}(A) \subset \mathrm{im}\delta_n.\end{aligned}$$

The rest of the proof is just using basic isomorphism theorems from group theory to show that $H_n(X, A) \cong \tilde{H}_n(X)$, for $n > 0$. When $n = 0$, $\mathrm{im}\delta_1^{rel} = {\mathrm{im}\delta_1 + C_0(A)}/{C_0(A)}$, and $\ker \delta_0^{rel} = {C_0(X)}/{C_0(A)}$. So,

$$H_0(X, A) \cong {C_0(X)}/{\mathrm{im}\delta_1 + C_0(A)} \cong {C_0(X)}/{\mathrm{im}\delta_1 \oplus \mathbb{Z}} = \tilde{H}_0(X).$$

□

In general, we have a short exact sequence of chain complexes

$$0 \to C_\bullet(A) \xrightarrow{i} C_\bullet(X) \xrightarrow{j} C_\bullet(X, A) \to 0,$$

which, as we know, gives us a long exact sequence of homology groups.

Theorem 4.13. *There is a long exact sequence of homology groups*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta_*} H_{n-1}(A) \to \cdots$$

The map $\delta_* : H_n(X, A) \to H_{n-1}(A)$ has a much nicer explanation. Elements of $\ker \delta_n^{rel}$ are mapped to $0 \mod C_{n-1}(A)$ by the boundary map. These are cycles in X whose boundaries lie in A. The map δ_* sends each cycle to its boundary.

Example 4.14. When A is a point, for all $n > 0$, $H_n(A) = 0$. This induces an isomorphism $H_n(X) \cong H_n(X, A)$ for all $n > 1$. When $n = 1$, we have

$$0 \xrightarrow{i_*} H_1(X) \xrightarrow{j_*} H_1(X, A) \xrightarrow{\delta_*} H_0(A)$$

This is why it is often helpful to consider reduced homologies. The corresponding short exact sequence of *augmented* chain complexes gives us a long exact sequence of the reduced relative homologies. This gives us the isomorphism $\tilde{H}_1(X) \cong \tilde{H}_1(X, A)$.

Remark 4.15 (Defining the reduced relative homology.)**.** The reduced relative homology $\tilde{H}_n(X, A)$ has two possible definitions. The standard definition (and the one used in these notes) is as the relative homology of the augmented chain complexes. Augment the chain complexes $C_\bullet(A)$ and $C_\bullet(X)$ as usual with the maps $C_0(A) \xrightarrow{\epsilon} \mathbb{Z} \to 0$ and $C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$. It is now clear that when we take the quotients of the chain groups, the resulting chain complex is exactly the same as $C_\bullet(X, A)$. The only difference is that the homology groups that appear in the long exact sequence are the reduced homology groups of X and A.

The groups $\tilde{H}_n(X, A)$ are *not* formed by augmenting the chain complex $\left({C_n(X)}/{C_n(A)}\right)$! Make sure you understand this difference.

Example 4.16 (The homologies of (D^n, S^{n-1}).). The long exact sequence for the reduced homologies yields isomorphisms $H_i(D^n, S^{n-1}) \to \tilde{H}_{i-1}(S^{n-1})$, since the reduced and non-reduced relative homology groups are the same. This tells us that $H_i(D^n, S^{n-1}) = \mathbb{Z}$ if $i = n$, and 0 otherwise.

To conclude this section, we can also look at induced homomorphisms between relative homology groups. For the rest of this section, (X, A) is a *pair* if $A \subset X$. A map $f : (X, A) \to (Y, B)$ is a *map of pairs* if $f : X \to Y$ is a map and $f(A) \subset B$. A map of pairs extends to a well-defined map on the quotient chain complex, so it induces a homomorphism $f_* : H_*(X, A) \to H_*(Y, B)$. We say two maps of pairs $f, g : (X, A) \to (Y, B)$ are homotopic if there is a homotopy $H : f \to g$ such that $H(A, t) \subset B$ for all $t \in [0, 1]$.

Proposition 4.17. *If $f, g : (X, A) \to (Y, B)$ are homotopic as maps of pairs, then they induce the same homomorphisms $H_*(X, A) \to H_*(Y, B)$.*

Exercise 8. Fill in the group-theoretic details in the proof of Proposition 4.12.

Exercise 9. Following Example 4.14, use the long exact sequence to compute $H_0(X, A)$ when A is a point.

4.3 Excision

The goal of this section is to prove the following result.

Theorem 4.18. *Suppose $A \subset X$ is a nonempty closed subspace that is a deformation retract of some neighborhood in X. Then, there is a long exact sequence of reduced homologies*[7]

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\delta_*} \tilde{H}_{n-1}(A) \to \cdots \to \tilde{H}_0(X/A) \to 0$$

But first, some applications!

Example 4.19 (The homologies of the spheres, using quotients.). For $n > 0$, let $X = D^n$ and $A = S^{n-1}$. A neighborhood of S^{n-1} in D^n deformation retracts onto it. The quotient ${D^n}/{S^{n-1}}$ is homeomorphic to S^n. If we know that the homology groups of S^1 are $H^1(S^1) = H^0(S^1) = \mathbb{Z}$, and $H^k(S^1) = 0$ for $k > 1$, we can use induction and the long exact sequence to compute the homology groups of S^n.

Example 4.20 (The incorrect homology of S^1.). Let $X = \mathbb{R}$ and $A = \mathbb{Z}$. Since $\mathbb{Z}$ has the discrete topology as a subspace of $\mathbb{R}$, it has the CW-complex structure of being a disjoint union of countably many points. So, $H_0(\mathbb{Z}) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$, and $H_n(\mathbb{Z}) = 0$ for $n > 0$. Since $\mathbb{R}$ is contractible, $H_0(\mathbb{R}) = \mathbb{Z}$ and $H_n(\mathbb{R}) = 0$ for $n > 0$. The quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1. From the long exact sequence of reduced homologies, $\tilde{H}_n(S^1) = 0$ for all $n \geq 2$. The last few terms in the long exact sequence are

$$0 \to \tilde{H}_1(S^1) \xrightarrow{\delta_*} \tilde{H}_0(\mathbb{Z}) \xrightarrow{i_*} \tilde{H}_0(\mathbb{R}) \xrightarrow{j_*} \tilde{H}_0(S^1) \to 0$$

Since $\tilde{H}_0(\mathbb{R}) = 0$, $\tilde{H}_0(S^1) = 0$. Also, this tells us that $\tilde{H}_1(S^1) \cong \tilde{H}_0(\mathbb{Z}) \cong \bigoplus_{\mathbb{Z}} \mathbb{Z}$.

The problem with the previous example is that there are two notions of quotients for $\mathbb{R}/\mathbb{Z}$. The one that yields the homeomorphism to S^1 is the quotient as a topological *group*. The other one, which is what we consider in this theorem, is the quotient as a topological *subspace*. We identify all the points of $\mathbb{Z}$ as a single point, so the quotient $\mathbb{R}/\mathbb{Z}$ is an infinite wedge sum of circles. There is nothing wrong with our computations in the previous example, only with our description of the quotient space $\mathbb{R}/\mathbb{Z}$.

The key to this proof is the *Excision theorem*, which is interesting in its own right.

Theorem 4.21.

[7] These are the usual reduced homologies of the quotient space X/A, not the trickery with reduced relative homologies.

1. *Given subspaces $Z \subset Y \subset X$ such that the closure of Z is contained in the interior of A, the inclusion map $(X - Z, A - Z) \to (X, A)$ induces isomorphisms on the relative homologies $H_*(X - Z, A - Z) \to H_*(X, A)$.*
2. *Given subspaces $A, B \subset X$ whose interiors cover X, the inclusion map $(B, A \cap B) \to (X, A)$ induces isomorphisms on the relative homologies $H_*(B, A \cap B) \to H_*(X, A)$.*

The two statements are equivalent by setting $B = X - Z$. Then, $\mathrm{cl}(Z) \subset \mathrm{int}(A)$ if and only if $\mathrm{int}(B)$ and $\mathrm{int}(A)$ cover X, since $\mathrm{int}(B) = X - \mathrm{cl}(Z)$.

Example 4.22 (The homologies of the spheres, using excision.). As always, our proof starts by induction, with the base case being the homologies of S^1. For $n > 1$, consider the pair (S^n, D^n), where D^n is embedded as the closed lower hemisphere. Let p be the south pole. By the Excision theorem, $H_*(S^n - \{p\}, D^n - \{p\}) \cong H_*(S^n, D^n) \cong \tilde{H}_*(S^n)$, since D^n deformation retracts to a point. However, the pair $(S^n - \{p\}, D^n - \{p\})$ is homotopy equivalent to the pair (D^n, S^{n-1}). We know that the homology groups of (D^n, S^{n-1}) are $H_i(D^n, S^{n-1}) = \mathbb{Z}$ if $i = n$, and 0 otherwise. So, $H_i(S^n) = \mathbb{Z}$ if $i = 0$ or n, and 0 otherwise.

Where did we need induction? We computed the relative homologies of (D^n, S^{n-1}) in Example 4.16 by induction. At the inductive step, we only needed to know the homologies of S^{n-1}, so there is no tautological argument here.

The proof of the Excision theorem is monstrous. The key tool is the following proposition.

Proposition 4.23. *Let $\mathcal{U}$ be an open cover of X, and let $C_n^{\mathcal{U}}(X)$ be the free abelian group generated by maps $\sigma : \Delta^n \to X$ whose images are contained in some set of $\mathcal{U}$. The inclusion $C_n^{\mathcal{U}}(X) \hookrightarrow C_n(X)$ induces isomorphisms on the homology groups $H_*^{\mathcal{U}}(X) \cong H_*(X)$.*

Now, to prove Theorem 4.18, we need to pass from the relative homology of (X, A) to the reduced homology of X/A. The quotient map $X \to X/A$ induces a map of pairs $(X, A) \to (X/A, *)$, since A is collapsed to a single point.

Corollary 4.24. *Suppose there is a subspace $B \subset X$ such that $\mathrm{cl}(A) \subset \mathrm{int}(B)$ and the inclusion $A \hookrightarrow B$ is a deformation retract. Then, the map $(X, A) \to (X/A, *)$ induces an isomorphism $H_*(X, A) \to \tilde{H}_*(X/A)$.*

Proof. See this link. □

Corollary 4.25. *Let X be the wedge sum of the spaces $(X_\alpha)_\alpha$ at the basepoints $x_\alpha \in X_\alpha$. If the pairs (X_α, x_α) are good, then the inclusion $\bigvee_\alpha X_\alpha \hookrightarrow \bigsqcup_\alpha X_\alpha$ s isomorphisms $H_*(\bigvee_\alpha X_\alpha) \cong \bigoplus_\alpha \tilde{H}_*(X_\alpha)$. (Using the expression for $H_*(\bigsqcup_\alpha X_\alpha)$ from Exercise 6.)*

Exercise 10. Write out the details of Example 4.19 to compute the homology groups of the spheres.

4.4 Mayer-Vietoris

The other long exact sequence is the *Mayer-Vietoris sequence*, the homological Van Kampen's theorem. Given a pair of subspaces $A, B \subset X$, let $C_n(A + B)$ be the chains in $C_n(X)$ that are sums of chains in $C_n(A)$ and $C_n(B)$. The boundary maps turn $C_\bullet(A + B)$ into a chain complex, and we get a short exact sequence of chain complexes

$$0 \to C_\bullet(A \cap B) \xrightarrow{i} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{j} C_\bullet(A + B) \to 0.$$

The map i sends $\sigma \in C_n(A \cap B)$ to the ordered pair $(\sigma, -\sigma) \in C_n(A) \oplus C_n(B)$. The map j sends $(\sigma_a, \sigma_b) \in C_n(A) \oplus C_n(B)$ to the sum $\sigma_a + \sigma_b$. As we now know, this induces a long exact sequence of homologies.

Theorem 4.26. *If $A, B \subset X$ such that $\mathrm{int}(A)$ and $\mathrm{int}(B)$ cover X, then there is a long exact sequence*

$$\cdots \to H_n(A \cap B) \xrightarrow{i_*} H_n(A) \oplus H_n(B) \xrightarrow{j_*} H_n(X) \xrightarrow{\delta_*} H_{n-1}(A \cap B) \to \cdots$$

The trick here is to use Proposition 4.23 so that the inclusion $C_n(A + B) \hookrightarrow C_n(X)$ induces isomorphisms $H_n(A + B) \cong H_n(X)$.

Example 4.27 (The homologies of the spheres, by Mayer-Vietoris). Once again, let $X = S^n$, and let A and B be slightly enlarged northern and southern hemispheres respectively. The spaces A and B are contractible, and $A \cap B$ is homotopic to S^{n-1}. In the Mayer-Vietoris sequence for reduced homologies, this yields isomorphisms $\tilde{H}_i(S^n) \cong \tilde{H}_i(S^{n-1})$.